Dispersion perturbation in a transfer line

The dispersion perturbation in a transfer line is given by

$$\Delta D(s) = \int_0^s \frac{1}{\rho(t)} \sqrt{\beta(t)\beta(s)} \sin(\mu(s) - \mu(t)) dt \tag{1}$$

where $\rho(s)$ is the radius of curvature. ρ can be approximated by

$$\rho = \frac{L}{\theta} \tag{2}$$

where θ is the kick and L is the element length.

The dispersion change induced at monitor l by a kick θ_j corrector j due to the beam excursion in quadrupole i is:

$$\Delta D_i^l = K_i L_i \Delta u_i \sqrt{\beta_l \beta_i} \sin(\mu_l - \mu_i) \theta_j \tag{3}$$

since for a beam with an offset of Δu_i in a quadrupole with strength K_i, ρ is:

$$\frac{1}{\rho} = K_i \Delta u_i \tag{4}$$

The beam offset Δu_i itself due to the kick is

$$\Delta u_i = \sqrt{\beta_i \beta_j} \sin(\mu_i - \mu_j) \theta_j \tag{5}$$

Putting it all together, the dispersion change at monitor l due to the kick from corrector j is:

$$\Delta D^l = \left\{ \sum_i K_i L_i \beta_i \sin(\mu_l - \mu_i) \sin(\mu_i - \mu_j) - \sin(\mu_l - \mu_j) \right\} \sqrt{\beta_l \beta_j} \theta_j \tag{6}$$

where the sum runs over all quadrupoles between the corrector and the monitor. The last term is the 'direct' term from the corrector kick itself.

Note :

- The sign in the last takes into account the fact that the sign definition for corrector and quadrupole kicks is opposite to the one needed for the definition of the dispersion (through ρ). This explains the $-$ sign in the second part of the equation.

- The equation is valid for the horizontal plane. For the vertical plane a $-$ sign must be inserted to take into account the strength sign.

Dispersion perturbation in a ring

The dispersion perturbation in a ring is given to first approximation by

$$\Delta D(s) = \frac{\sqrt{\beta(s)}}{2\sin(\pi Q)} \oint_0^C \frac{\sqrt{\beta(t)}}{\rho(t)} \cos(|\mu(s) - \mu(t)| - \pi Q) dt \tag{7}$$

where $\rho(s)$ is the radius of curvature and Q is the tune. This expression does not include the higher order terms from sextupoles that will be discussed later.

The dispersion change induced at monitor l by a kick θ_j corrector j due to the beam excursion in quadrupole i is:

$$\Delta D_i^l = K_i L_i \Delta u_i \frac{\sqrt{\beta_l \beta_i}}{\sin(\pi Q)} \cos(|\mu_l - \mu_i| - \pi Q)\theta_j \tag{8}$$

since for a beam with an offset of Δu_i in a quadrupole with strength K_i, ρ is:

$$\frac{1}{\rho} = K_i \Delta u_i \tag{9}$$

The beam offset Δu_i itself due to the kick is

$$\Delta u_i = \frac{\sqrt{\beta_i \beta_j}}{\sin(\pi Q)} \cos(|\mu_i - \mu_j| - \pi Q)\theta_j \tag{10}$$

The sextupoles also contribute to the dispersion to first order in the orbit perturbation. This is due to the that the the dispersion may be defined as

$$D_u = \lim_{\delta \to 0} \frac{u(\delta) - u(0)}{\delta} \tag{11}$$

where $u(\delta)$ is the closed orbit as a function of the momentum offset δ. The effect of the sextupoles is proportional to

$$\propto K_2 x^2 \propto K_2(x(0) + D_x\delta)^2 \propto K_2 x(0)^2 + 2K_2 D_x x(0)\delta + D_x^2\delta^2 \tag{12}$$

and

$$\propto K_2 xy \propto K_2(x(0) + D_x\delta)y(0) \propto K_2 x(0)y(0) + K_2 D_x y(0)\delta \tag{13}$$

The effect of the sextupole is obtained by replacing the term K_i in the equation for the quadrupoles by $-K_{2,i}D_{x,i}$, see for example the SLAC paper by R. Z. Liu (SLAC/AP-14, 1984).

Putting it all together, the dispersion change at monitor l due to the kick from corrector j is:

$$\begin{aligned}\Delta D^l &= \{\sum_i \frac{K_i L_i \beta_i}{4\sin(\pi Q)^2}\cos(|\mu_l-\mu_i|-\pi Q)\cos(|\mu_i-\mu_j|-\pi Q) \\ &\quad -\sum_m \frac{K_{2,m}D_{x,m}L_m\beta_m}{4\sin(\pi Q)^2}\cos(|\mu_l-\mu_m|-\pi Q)\cos(|\mu_m-\mu_j|-\pi Q) \\ &\quad -\frac{\cos(|\mu_l-\mu_j|-\pi Q)}{\sin(\pi Q)}\}\sqrt{\beta_l\beta_j}\theta_j \qquad (14)\end{aligned}$$

where the sums run over all quadrupoles (i) and sextupoles (m). The last term is the 'direct' term from the corrector kick itself.

Momentum compaction factor in a ring

The momentum compaction factor α_c is given by:

$$\alpha_c = \frac{1}{C}\oint \frac{D_x(s)ds}{\rho(s)} \qquad (15)$$

becomes

$$\alpha_c = \frac{1}{C}\sum_i D_{x,i}\theta_i \qquad (16)$$

where the sum runs over all bending magnets and θ_i is the deflection at bend number i.